Solutions to Exam 4

---Which pair of planes are parallel?
Solution: Both planes are written in the general form ax + by + cz = d where n = <a, b, c>. Two planes with normal vectors n_1 and n_2 are parallel if n_1 is a scalar multiple of n_2. The first plane has normal: <1,2,-3> and the second plane has normal <2, 4, -6>. $2n_1 = n_2$

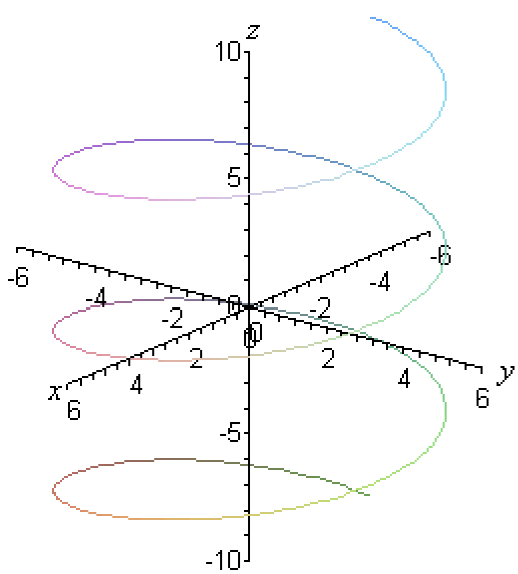
$$x + 2y - 3z = 4 \ \ and \ \ 2x + 4y - 6z = 3$$

---Find a vector equation for the line through the point $(-10, 8, 4)$ and parallel to the vector $< 2, 4, -7 >$
solution: use the vector equation **r**(t) = $\mathbf{r_0}$ + t**v** where $\mathbf{r_0}$ = < -10, 8, 4 > (or -10i + 8j + 4k) and **v** = direction vector = < 2, 4, -7 > (or 2i + 4j – 7k)

$$r(t) = (-10i + 8j + 4k) + t(2i + 4j - 7k)$$

---Which equation matches the graph? Ans: helix

$$\mathrm{r(t)} = < \cos(4t),\ \sin(4t),\ t >$$



--- Find the domain of the following vector function. Solution: the domain is the intersection of the 3 domains:

$t - 4 \geq 0$ and therefore the domain is $t \geq 4$
$10 - t \geq 0$ and therefore the domain is $t \leq 10$
t^9: domain is all real numbers
intersection: $4 \leq t \leq 10$

$$r(t) = \left\langle t^9,\ \sqrt{t-4},\ \sqrt{10-t} \right\rangle$$

---Find the unit tangent vector to the graph of $r(t) = < t^2, t^3 >$ at the point where t = 1.

Answer: use the formula: $\dfrac{r'(t)}{\| r'(t) \|}$ where $r'(t) = < 2t, 3t^3 >$ and $r'(1) = < 2, 3 >$

and $\|r'(1)\| = \sqrt{2^2 + 3^2} = \sqrt{13}$

$$< \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} >$$

---Which of the following curves is smooth?

Solution:

$r(t) = < t^2, t^8, t^5>$ $r'(t) = <2t, 8t^7, 5t^4>$ $r'(t) = 0$ when $t = 0$

$r(t) = < \cos(t), \sin(t) - t, 2\pi >$ $r'(t) = <-\sin t, \cos t - 1, 0>$; $r'(t) = 0$ when $t = 0$

$r(t) = < t^7 + t, t^2, t^6>$ $r'(t) = < 7t^6 + 1, 2t, 6t^5 >$ Here, $r'(t)$ can never be the zero vector, therefore $r(t)$ is smooth

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---Use the following formula to find the distance between the two parallel planes given by

$3x - y + 2z - 6 = 0$ and $6x - 2y + 4z + 4 = 0$

Formula: $$D = \frac{| ax_0 + by_0 + cz_0 + d |}{\sqrt{a^2 + b^2 + c^2}}$$

Use a point from the first plane [such as (2,0,0)] as (x_0, y_0, z_0) and the coefficients from the second plane

a = 6, b = -2, c = 4 and d = 4. Plug into formula. Ans: $\dfrac{8}{\sqrt{14}}$

----Use the formula V = $|u \bullet (v \otimes w)|$ to find the volume of the parallelepiped constructed by $u = <3,2,1>, \quad v = <-1,3,0>, \quad w = <2,2,5>$

solution: use the scalar triple product (see pp. 818-9 in the text) Answer: 47

---Find an equation of the plane through the points P_1 (1,2,-1), P_2 (2,3,1), and P_3 (3,-1,2).

Find the vector $\overrightarrow{P_2P_1} = <1,1,2>$

$\overrightarrow{P_3P_1} = <2,-3,3>$

Take the cross-product of these two vectors to get the normal: <9, 1, -5>

Use one of the original points (I'll use P_1) and the standard equation for a plane $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$:

$9(x - 1) + 1(y - 2) - 5(z + 1) = 0$

---Find the arc length of the portion of the circular helix:

x = cos(t), y = sin(t), z = t from t = 0 to t = π

Use the formula: $\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$ plug in:

$$\int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt = \int_0^\pi \sqrt{1+1}\, dt \quad (\text{since } \sin^2 t + \cos^2 t = 1)$$

final answer: $\sqrt{2}\pi$

---Find **r**(t) given that **r**'(t) = < 3, 2t > and **r**(1) = < 2, 5 >

Solution: r(t) = $\int <3,2t> dt \;=\; <3t + C_1, t^2 + C_2>$

When t = 1:

$<2,5> \;=\; <3(1) + C_1, (1)^2 + C_2>$ therefore: $3 + C_1 = 2$ and $C_1 = -1$

$1 + C_2 = 5$ and $C_2 = 4$

Ans: $r(t) = <3t - 1, t^2 + 4>$

---Using the formula learned in class: $r(t) = (v_0 \cos\theta)ti + [h + (v_0 \sin\theta)t - \frac{1}{2}gt^2]j$, solve the following motion problem, letting g = 32 feet per second per second. [I suggest that you do *not* approximate any square roots by a decimal and just take your time working out the numbers.]:

A shell, fired from a cannon, has a muzzle speed (the speed as it leaves the barrel) of 800 ft/s. The barrel makes an angle of 45º with the horizontal and, for simplicity, the barrel opening is assumed to be at ground level.

a) Use the given formula to write *parametric equations* for the shell's trajectory.

$$x(t) = 800 \cdot \cos(45)t$$

$$x(t) = 800 \cdot \frac{\sqrt{2}}{2} t$$

$$x(t) = (400\sqrt{2})t \text{ (ans.)}$$

$$y(t) = 0 + (800 \sin 45)t - \frac{1}{2} \cdot 32t^2$$

$$y(t) = (400\sqrt{2})t - 16t^2 \text{ (ans.)}$$

b) Using your equation(s) from part a), how high does the shell rise?

$$y'(t) = (400\sqrt{2}) - 32t$$

$$0 = (400\sqrt{2}) - 32t$$

$$t = \frac{400\sqrt{2}}{32} \qquad t = \frac{25}{2}\sqrt{2} \text{ (max height occurs at this time)}$$

Now plug this time back into the original y(t) equation to find the maximum height

$$y(\frac{25}{2}\sqrt{2}) = 400\sqrt{2}(\frac{25}{2}\sqrt{2}) - 16(\frac{25}{2}\sqrt{2})^2$$

$$y(\frac{25}{2}\sqrt{2}) = 200(2)(25) - 16(\frac{625 \cdot 2}{4})$$

$$y(\frac{25}{2}\sqrt{2}) = 10{,}000 - 8(625) = \underline{5000} \text{ (ans)}$$